GEOMETRY OF GR

$k = \bar{k}$ FOR TODAY

PRELIM RMKS:

1) $n=1$ $GL_1 = \mathbb{G}_m$. $L\mathbb{G}_m$ AND GR ARE NOT REDUCED.

CAN STILL CONCLUDE THAT

$$(\mathrm{GR})_{\mathrm{RED}} = \mathbb{Z}$$

FROM $\mathrm{GR} = \operatorname{colim}_N \mathrm{GR}^{(N)}$

$$\left(\begin{array}{l} \text{IF } R \text{ IS A FIELD, THEN} \\ \quad \mathrm{GR}^{(N)}(R) = \{ t\text{-STABLE SUBSPACES OF } a_{-N}t^{-N} + a_{-N+1}t^{-N+1} \\ \qquad \cdots + a_{N-1}t^{N-1} (\mathrm{mod}\ t^N) \} \\ t \text{ ACTS AS A JORDAN BLOCK} \\ \Longrightarrow 2N+1 \text{ SUBSP'S} \end{array} \right)$$

2) ARB n: HAVE $\det : GL_n \longrightarrow GL_1$, INDUCES

$$\mathrm{GR}_{GL_n} \longrightarrow \mathrm{GR}_{GL_1}$$

$$\Lambda \longmapsto \det_{R[[t]]}(\Lambda)$$

THIS MAP IDENTIFIES CONNECTED COMPS

$$\Longrightarrow \pi_0(\mathrm{GR}_{GL_n}) = \mathbb{Z}$$

3) IN GENERAL, IF G IS A SPLIT RED GP / k, THEN WE HAVE

$$\underset{\substack{\cup\!\!\!\mid \\ [LG/L^+G]}}{\pi_0(Gr_G)} = \pi_1(G)$$

(IDEA: IF $k = \mathbb{C}$, $Gr_G \cong \Omega K$

Ω: LOOP SPACE

K: MAX'L COMPACT OF $G(\mathbb{C})$

$$\begin{aligned} \pi_0(Gr_G) &= \pi_0(\Omega K) \\ &= \pi_1(K) \\ &= \pi_1(G) \end{aligned}$$
)

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SCHUBERT VARIETIES

USED FOR GEOM. SATAKE

NOTATION $\mathcal{O} = k[[t]]$, $F = k((t))$

$$T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \subset GL_n$$

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) = \mathbb{Z}^n$$

IF $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n$, THEN

$$t^\mu := \mu(t) = \begin{pmatrix} t^{\mu_1} & & & \\ & t^{\mu_2} & & \\ & & \ddots & \\ & & & t^{\mu_n} \end{pmatrix} \in LT \hookrightarrow LGL_n$$

CHOICE OF B GIVES

$$X_*(T)^+ = \{\mu = (\mu_1, \mu_2, \dots, \mu_n) : \mu_1 \geq \mu_2 \geq \dots \geq \mu_n\} =: \mathbb{Z}^n_+$$

GET A PARTIAL ORDER ON $X_*(T)$:

$\mu, \lambda \in X_*(T)$

$\mu \geq \lambda \iff \mu - \lambda = \sum_{i=1}^{n-1} c_i \alpha_i^\vee$

$c_i \in \mathbb{Z}_{\geq 0}$, $\alpha_i^\vee = (0, \ldots, 0, \underset{i}{1}, \underset{i+1}{-1}, 0, \ldots, 0)$

<u>FACT</u> (CARTAN DECOMP)

$$GL_n(F) = \bigsqcup_{\mu \in \mathbb{Z}^n_+} GL_n(\mathcal{O}) t^\mu GL_n(\mathcal{O})$$

PF FOLLOWS FROM THM ON ELEMENTARY DIVISORS

FUN EXERCISE ($n=2$)

<u>RMK</u> WORKS FOR ANY FIELD EXT OF k

<u>DEF</u> LET $\mu \in \mathbb{Z}^n_+$. DEFINE

$Gr_\mu = L^+GL_n \cdot t^\mu \subset Gr$ "SCHUBERT CELL"

$Gr_{\leq \mu} = \overline{Gr_\mu}$ "SCHUBERT VARIETY"

+ EQUIP THESE W/ REDUCED SUBSCHEME STR.

<u>PROP</u> 1) $(Gr)_{RED} = \operatorname{colim}_\mu Gr_{\leq \mu}$ (L^+GL_n-STABLE)

2) Gr_μ IS SMOOTH, Q. PROJ, FINITE TYPE $/k$

3) IF $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n_+$, THEN

$$\mathrm{DIM}(\mathrm{Gr}_\mu) = (n-1)\mu_1 + (n-3)\mu_2 + \dots + (1-n)\mu_n$$
$$\Big(= \sum_{\alpha \text{ pos root}} \langle \alpha, \mu \rangle = \langle 2\rho, \mu \rangle \Big)$$

4) $\mathrm{Gr}_{\leq \mu}$ IS PROJ, IRRED AND

$$\mathrm{Gr}_{\leq\mu} = \bigsqcup_{\lambda \leq \mu} \mathrm{Gr}_\lambda$$

5) $\mathrm{Gr}_{\leq\mu}$ IS NORMAL, GORENSTEIN, AND HAS RAT'L SINGULARITIES. (FROB SPLIT IF CHAR$(k) = p$)

PF 1) FOLLOWS FROM CARTAN DECOMP

2) + 3) STAB IN L^+GL_n OF $t^\mu \in \mathrm{Gr}$ IS
$$L^+GL_n \cap t^\mu L^+GL_n t^{-\mu}$$

$\rightsquigarrow$ GET A LOCALLY CLOSED EMBEDDING

$$L^+GL_n / (L^+GL_n \cap t^\mu L^+GL_n t^{-\mu}) \longrightarrow LGL_n / L^+GL_n$$
$$g \longmapsto g \cdot t^\mu$$

IMAGE IS Gr_μ $\Longrightarrow$ Gr_μ IS SMOOTH

GET THAT TGT SPACE OF LHS IS
$$\mathfrak{gl}_n(\mathcal{O}) / \mathfrak{gl}_n(\mathcal{O}) \cap t^\mu \mathfrak{gl}_n(\mathcal{O}) t^{-\mu}$$
$$= \mathfrak{gl}_n(\mathcal{O}) \Big/ \begin{pmatrix} \mathcal{O} & & t^{\mu_i - \mu_j}\mathcal{O} \\ & \ddots & \\ \mathcal{O} & & \mathcal{O} \end{pmatrix} \quad (i,j)^{\text{TH}} \text{ PLACE}$$

$$\Longrightarrow \mathrm{DIM} = \sum_{i<j} (\mu_i - \mu_j) = (n-1)\mu_1 + (n-3)\mu_2 + \dots$$

4) NEED TO SHOW IF $\lambda \in \mathbb{Z}^n_+$, $\lambda < \mu$, THEN $Gr_\lambda \subset \overline{Gr_\mu}$

$\lambda < \mu \Rightarrow \exists$ POS COROOT $\alpha^\vee$ ST $\lambda \leq \mu - \alpha^\vee < \mu$

SUFFICES TO SHOW $t^{\mu-\alpha^\vee} \in \overline{Gr_\mu}$

<u>IDEA</u> WANT A MAP

$$\mathbb{P}^1 \longrightarrow Gr$$

ST
- IMAGE OF $\mathbb{A}^1 \subset Gr_\mu$
- IMAGE OF $\infty = t^{\mu-\alpha^\vee}$

REDUCES TO AN SL_2 CALCULATION

WRITE $\alpha^\vee = (0, \ldots, 0, \underset{i}{1}, \ldots \underset{j}{-1}, \ldots 0)$

+ SET
$$\begin{aligned} m &= \mu_i - \mu_j - 1 \\ &= \langle \alpha, \mu \rangle - 1 \quad (\alpha \text{ IS THE ROOT CORR. TO } \alpha^\vee) \\ &\geq 0 \quad (\text{B/C } \lambda \leq \mu - \alpha^\vee < \mu) \end{aligned}$$

DEFINE

$$K_m = \begin{pmatrix} \mathcal{O} & t^m \mathcal{O} \\ t^{-m}\mathcal{O} & \mathcal{O} \end{pmatrix} \subset L\,SL_2$$

$$J_m = \begin{pmatrix} \mathcal{O}^\times & t^{m+1}\mathcal{O} \\ t^{-m}\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \subset LSL_2$$

THEN
$$K_m / J_m \cong SL_2 / \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \cong \mathbb{P}^1$$
$$= \left\{ \begin{pmatrix} & -t^m \\ t^{-m} & \end{pmatrix}, \begin{pmatrix} 1 & t^m x \\ & 1 \end{pmatrix} : x \in \mathbb{A}^1 \right\}$$

COSET REPS

LET $i_\alpha : SL_2 \longrightarrow GL_n$ BE THE MORPHISM ASS'D TO α, IE

$$i_\alpha : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \qquad\quad i \qquad\quad j \\ \begin{pmatrix} 1 & & & & \\ & a & & b & \\ & & \ddots & & \\ & c & & d & \\ & & & & 1 \end{pmatrix} \end{array}$$

WE HAVE

- $i_\alpha(J_m) \subset t^\mu L^+GL_n t^{-\mu} = \mathrm{STAB}_{LGL_n}(t^\mu)$
- $i_\alpha \begin{pmatrix} 1 & t^m x \\ & 1 \end{pmatrix} \in L^+GL_n \setminus t^\mu L^+GL_n t^{-\mu}$
- $i_\alpha \begin{pmatrix} & -t^m \\ t^{-m} & \end{pmatrix} = i_\alpha \left(\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -t^{m+1} \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ t^{-m-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & -t^{m+1} \\ & 1 \end{pmatrix}}_{g} \right)$

$$= t^{-\alpha^\vee} \underbrace{i_\alpha(g)}_{\in\, t^\mu L^+GL_n t^{-\mu}}$$

SO, WE GET

- $i_\alpha(K_m) \cdot t^\mu \cong K_m / J_m \cong \mathbb{P}^1$
- $\mathbb{A}^1 \cong i_\alpha \begin{pmatrix} 1 & t^m x \\ & 1 \end{pmatrix} \cdot t^\mu \subset GR_\mu$
- $\mathbb{P}^1 \setminus \mathbb{A}^1 = \{\infty\} = i_\alpha \begin{pmatrix} & -t^m \\ t^{-m} & \end{pmatrix} \cdot t^\mu = t^{\mu - \alpha^\vee}$

$\square$

LOOSE ANALOGY

GR	$\longleftrightarrow$	GL_n/B FLAG VAR
$\mathbb{Z}^n_+$, $\leq$ DOM ORD	$\longleftrightarrow$	S_n, $\leq$ BRUHAT ORDER
GR_μ	$\longleftrightarrow$	BwB/B SCHUBERT CELLS
$GR_{\leq\mu}$	$\longleftrightarrow$	$\overline{BwB}/B$ SCHUBERT VAR

EXAMPLES

① IF $\mu = (a, a, \dots, a) \in \mathbb{Z}^n_+$ THEN t^μ IS CENTRAL

$$\Longrightarrow \quad GR_\mu = GR_{\leq\mu} = \{t^\mu\} = *$$

② LET $n=2$, $\mu = (1,0) \in \mathbb{Z}^n_+$. THEN μ IS MINIMAL FOR $\leq$.

$$\Longrightarrow \quad GR_{\leq\mu} = GR_\mu \cong L^+GL_2 / L^+GL_2 \cap \begin{pmatrix} t & \\ & 1 \end{pmatrix} L^+GL_2 \begin{pmatrix} t^{-1} & \\ & 1 \end{pmatrix}$$

$$\cong L^+GL_2 / \begin{pmatrix} \mathcal{O}^\times & t\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times \end{pmatrix}$$

$\cong \mathbb{P}^1$

MORE GENERALLY:

PROP LET n BE ARBITRARY, $\mu \in \mathbb{Z}^n_+$ MINUSCULE

($\langle \alpha, \mu \rangle \in \{0,1\}$ FOR ANY POS ROOT α;
FOR GL_n: $\mu = (\underbrace{1,1,\dots,1}_{i}, \underbrace{0,0,\dots 0}_{n-i})$)

THEN

$$Gr_{\leq\mu} = Gr_\mu = G/P_\mu = Gr(n-i,n)$$

WHERE

$$P_\mu = \begin{array}{c} \\ i\{ \\ n-i\{ \end{array} \begin{array}{c} \begin{array}{cc} \overbrace{\quad\;}^{i} & \overbrace{\quad\;}^{n-i} \end{array} \\ \left(\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right) \end{array}$$

③ $n=2$. $\mu = (a,b)$, $a > b$. THEN

$$Gr_\mu \cong L^+GL_2 / L^+GL_2 \cap \begin{pmatrix} t^a & \\ & t^b \end{pmatrix} L^+GL_2 \begin{pmatrix} t^{-a} & \\ & t^{-b} \end{pmatrix}$$

$$\cong L^+GL_2 / \begin{pmatrix} \mathcal{O}^\times & t^{a-b}\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times \end{pmatrix}$$

$$\longrightarrow L^+GL_2 / \begin{pmatrix} \mathcal{O}^\times & t\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times \end{pmatrix} \cong \mathbb{P}^1$$

FIBERS ARE $\cong t\mathcal{O}/t^{a-b}\mathcal{O} \cong \mathbb{A}^{a-b-1}$

MORE GENERALLY

PROP LET n BE ARB, $\mu = (\mu_1, \mu_2, \ldots \mu_n) \in \mathbb{Z}^n_+$.
Gr_μ IS THE TOTAL SPACE OF AN AFFINE BUNDLE OVER G/P_μ, WHERE

$$P_\mu = \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \boxed{*} & \\ * & & & \boxed{*} \end{pmatrix}$$

(NEW BLOCK WHEN $\mu_i > \mu_{i+1}$)

④ SPECIAL CASE: $n=2$, $\theta = \mu = (1,-1) \in \mathbb{Z}^n_+$

Gr_θ IS THE TOTAL SPACE OF

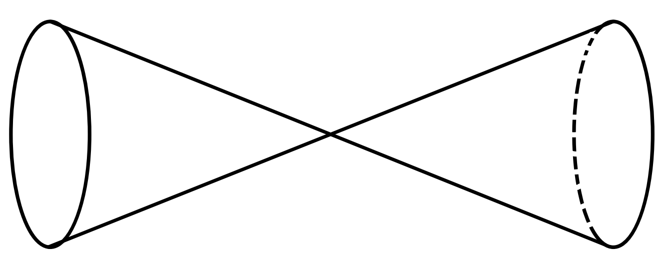
$$\mathcal{O}_{\mathbb{P}^1}(2) \cong G \times^{B^-} \theta$$

$\left(\begin{array}{l} \text{CONFLATING } \theta \in X_*(T) \text{ w/ } \theta \in X^*(T) \\ \text{GIVES } \theta : B^- \twoheadrightarrow T \longrightarrow \mathbb{G}_m \end{array} \right)$

THIS GIVES AN EMBEDDING

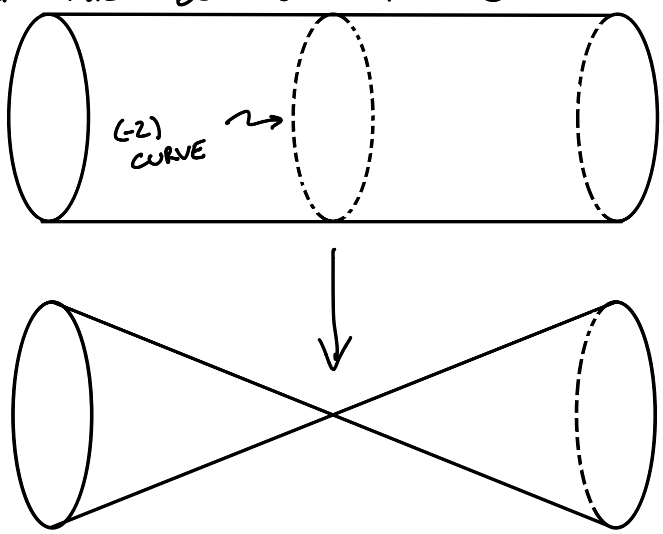
$$\begin{aligned} G/B^- \cong \mathbb{P}^1 &\hookrightarrow \mathbb{P}^2 \\ [x:y] &\longmapsto [x^2 : xy : y^2] \end{aligned}$$

AFFINE CONE OVER IMAGE IS $XZ - Y^2$ AND $Gr_{\leq\theta} = Gr_\theta \sqcup Gr_{(0,0)}$ IS THE PROJ CONE OVER $XZ - Y^2$:



$Gr_{\leq\theta}$

TAKING THE BLOW UP AT 0 GIVES A RESN



$$\widetilde{Gr}_{\le\theta} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1})$$

$$\downarrow$$

$$Gr_{\le\theta}$$

MORE GENERALLY:

<u>PROP</u> LET n BE ARBITRARY, $\theta = (1, 0, \dots, 0, -1)$, LET $\mathcal{L}$ BE THE VERY AMPLE LINE BUNDLE $G \times^{P_\theta} \theta$ OVER G/P_θ. THEN

Gr_θ = TOTAL SPACE OF $\mathcal{L}$

$Gr_{\le\theta} = Gr_\theta \sqcup Gr_0$ IS THE PROJ CONE OVER THE EMBEDDING OF G/P_θ GIVEN BY $\mathcal{L}$

$\widetilde{Gr}_{\le\theta} = \mathbb{P}(\mathcal{L}\oplus\mathcal{O})$ IS A RESN OF SINGULARITIES.

More Geometry

Recall: $\mathrm{Gr}_\mu = L^+GL_n \cdot t^\mu$, $\mathrm{Gr}_{\le\mu} = \overline{\mathrm{Gr}_\mu}$

Examples

⑤ Suppose $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n_+$ w/ $\mu_n \ge 0$

Then μ is a partition of $N = \sum_i \mu_i$

Lemma

$$\mathrm{Gr}_\mu(k) = \left\{ \Lambda \subset \Lambda_0 : \begin{array}{l} \bullet\ \dim(\Lambda_0/\Lambda) = N \\ \bullet\ t \circlearrowright \Lambda_0/\Lambda \text{ is nilpotent} \\ \quad \text{of type } \mu \end{array} \right\}$$

Can use this to recover minuscule case

Ex $n=2$, $\mu = (2,0)$

$$\mathrm{Gr}_\mu(k) = \left\{ \Lambda \subset \Lambda_0 : \begin{array}{l} \bullet\ \dim(\Lambda_0/\Lambda) = 2 \\ \bullet\ t \text{ acts as } \square\square \end{array} \right\}$$

Fun exercise: show this gives a line bundle / $\mathbb{P}^1$

$$\mathrm{Gr}_{\le\mu}(k) = \mathrm{Gr}_\mu(k) \sqcup \mathrm{Gr}_{(1,1)}(k)$$
$$= \{ \Lambda \subset \Lambda_0 : \dim(\Lambda_0/\Lambda) = 2 \} \subset \mathrm{Gr}(2,4)$$

DEFINE

$$\widetilde{GR}_{\leq\mu}(k) = \{ \Lambda \subset \Lambda' \subset \Lambda_0 : \dim(\Lambda_0/\Lambda') = \dim(\Lambda'/\Lambda) = 1 \}$$

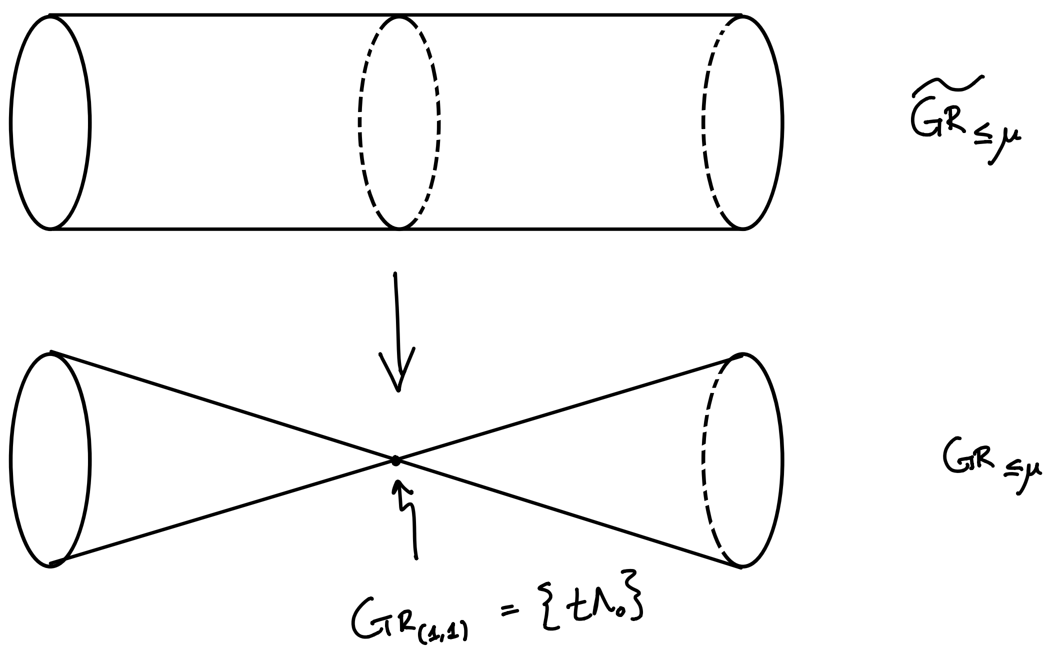
THIS IS A $\mathbb{P}^1$ BUNDLE OVER $\mathbb{P}^1$

HAVE A MAP

$$\begin{aligned} \widetilde{GR}_{\leq\mu} &\longrightarrow GR_{\leq\mu} \\ (\Lambda, \Lambda') &\longmapsto \Lambda \end{aligned}$$

THIS IS THE SAME RESN OF SINGS FROM LAST TIME

(IF $\Lambda \neq t\Lambda_0$, $\exists!$ Λ' ST $\Lambda \subset \Lambda' \subset \Lambda_0$.
IF $\Lambda = t\Lambda_0$, THEN $\exists$ $\mathbb{P}^1$ OF Λ''s)



GIVEN $\mu_1, \mu_2 \in \mathbb{Z}^n_+$, WE CAN DEFINE A "CONVOLUTION VERSION" OF SCHUBERT VARS

DEF

$$Gr_{\leq \mu_1} \tilde{\times} Gr_{\leq \mu_2} = \Big\{ (\Lambda_1, \Lambda_2, \beta_1, \beta_2) :$$

$$\begin{aligned}
&\cdot\ \Lambda_i \text{ F.G. PROJ } R[\![t]\!]\text{-MODS} \\
&\cdot\ \beta_1 : \Lambda_1[1/t] \xrightarrow{\sim} R(\!(t)\!)^n \\
&\cdot\ \beta_2 : \Lambda_2[1/t] \xrightarrow{\sim} \Lambda_1[1/t] \\
&\cdot\ \text{INV}(\beta_i) \leq \mu_i \Big\}
\end{aligned}$$

$=: Gr_{\leq \mu_\bullet}$ ← NOTATION

$\text{INV}(\beta) \in \mathbb{Z}^n_+$ MEASURES RELATIVE POS. OF β:

$$\begin{tikzcd}
\beta : \Lambda[1/t] \arrow[r, "\sim"] \arrow[d, "\phi"', "\wr"] & \Lambda'[1/t] \arrow[d, "\wr"', "\phi'"] \\
R(\!(t)\!)^n \arrow[r, dashed] & R(\!(t)\!)^n
\end{tikzcd}$$

(ϕ, ϕ' INDUCED FROM $\Lambda, \Lambda' \xrightarrow{\sim} R[\![t]\!]^n$)

$\dashrightarrow$ GIVES AN ELT OF $L\,GL_n(R)$

AMBIGUITY OF ϕ, ϕ' $\longleftrightarrow$ LEFT + RIGHT MULT BY $L^+GL_n(R)$

IF R IS A FIELD THIS GIVES AN ELT $\text{INV}(\beta)$ OF $\mathbb{Z}^n_+$ (BY CARTAN DECOMP)

IN GENERAL, WE WRITE $\text{INV}(\beta) \leq \mu$ IF $\text{INV}(\beta_{k(x)}) \leq \mu$ $\forall x \in \text{SPEC}(R)$

FOR $\mu \in \mathbb{Z}^n_+$, LET $LG_{\leq \mu}$ AND LG_μ DENOTE THE PREIMAGES OF $Gr_{\leq \mu}$ AND Gr_μ IN LG $(G = GL_n)$
(THESE ARE SCHEMES, w/ REDUCED SUBSCHEME STR)

FACT $\quad Gr_{\leq \mu_1} \tilde{\times} Gr_{\leq \mu_2} \cong LG_{\leq \mu_1} \times^{L^+GL_n} Gr_{\leq \mu_2}$

ALSO HAVE CONVOLUTION MAP

$$Gr_{\leq \mu_1} \tilde{\times} Gr_{\leq \mu_2} \longrightarrow Gr_{\leq \mu_1 + \mu_2}$$
$$(\Lambda_1, \Lambda_2, \beta_1, \beta_2) \longmapsto (\Lambda_2, \beta_1 \circ \beta_2)$$

("DEMAZURE / BOTT–SAMELSON" "RESN")

EX $\quad n = 2$, $\mu_1 = (1,0)$, $\mu_2 = (0,-1)$
THEN $\mu_1 + \mu_2 = \theta = (1,-1)$ AND

$$\begin{array}{ccc} Gr_{\leq \mu_1} \tilde{\times} Gr_{\leq \mu_2} & \longrightarrow & Gr_{\leq \theta} \\ \text{\rotatebox{90}{$\cong$}} & & \\ \mathbb{P}^1 \tilde{\times} \mathbb{P}^1 & & \end{array}$$

THIS IS THE SAME RESN OF SINGS AS BEFORE
[EX. 2.1.18]

OPPOSITE SCHUBERT "VARIETIES"

DEF LET G BE A LINEAR ALG GP $/k$

DEFINE THE PRESHEAF L^-G BY

$$L^-G(R) = G(R[t^{-1}])$$

"NEGATIVE LOOP GP"

INTUITION L^-G SHOULD BE LIKE AN OPPOSITE PARABOLIC SUBGP

EX $G = \mathbb{G}_m$, THEN

$$(L^-\mathbb{G}_m)(R) = R[t^{-1}]^\times$$

$$= \{a_0 + a_1 t^{-1} + \dots + a_n t^{-n} : a_0 \in R^\times,\ a_i \text{ NILPOTENT FOR } i \geq 1\}$$

$$= \left(\operatorname{colim}_i \operatorname{SPEC}\left(k[T_0^{\pm 1}, T_1, \dots, T_i] \big/ (T_1^i, T_2^i, \dots T_i^i)\right)\right)(R)$$

LEMMA L^-GL_n IS REP'D BY AN IND-SCHEME, AND IT IS A SUB-IND-SCHEME OF LGL_n

SKETCH GL_n IS DEFINED BY $y \det(x_{i,j}) = 1$

DEFINE $X_{k,\ell} = \{ \mathcal{F}(t^{-1}) = A^0_{i,j} + A^1_{i,j} t^{-1} + \dots + A^k_{i,j} t^{-k},$
$(k,\ell \geq 0)$
$g(t^{-1}) = b_0 + b_1 t^{-1} + \dots + b_\ell t^{-\ell} :$
$g(t^{-1}) \det(\mathcal{F}(t^{-1})) = 1 \}$

EXPANDING $g(t^{-1}) \det(\mathcal{F}(t^{-1}))$ GIVES RELS ON COEFFS

$\Rightarrow$ $X_{k,\ell}$ IS REP'BLE BY AN AFFINE SCHEME

AND $L^- GL_n = \operatorname{colim}_{k,\ell} X_{k,\ell}$ $\square$

$$\boxed{\begin{array}{l}\underline{\text{FACT}}\ (\text{BIRKHOFF DECOMP}) \\ GL_n(F) = \coprod_{\mu \in \mathbb{Z}^n_+} GL_n(k[t^{-1}])\, t^{\mu}\, GL_n(\mathcal{O})\end{array}}$$

(COMPARE: $GL_n = \coprod_{w \in S_n} B^- w B$)

SKETCH GIVEN $g \in GL_n(F)$, CAN MULTIPLY BY PERM. MATRICES ON RIGHT SO $\mathrm{val}(g_{1,1}) = \min(\mathrm{val}(g_{1,j}))$

ROW OPS ON RIGHT $\Rightarrow$ $\exists\, k \in GL_n(\mathcal{O})$ S.T

$$gk = \left(\begin{array}{c|ccc} g_{1,1} & 0 & \text{---} & 0 \\ \hline * & & * & \end{array}\right)$$

BY INDUCTION ON RANK, $\exists\ h \in GL_{n-1}(k[t^{-1}]),\ k' \in GL_{n-1}(\mathcal{O})$ ST

$$\begin{pmatrix} 1 & \\ & h \end{pmatrix} gk \begin{pmatrix} 1 & \\ & k' \end{pmatrix} = \left(\begin{array}{c|cccc} g_{1,1} & 0 & \text{---} & & 0 \\ \hline * & t^{c_2} & & & 0 \\ | & & \ddots & & \\ | & & & \ddots & \\ * & 0 & & & t^{c_n} \end{array}\right)$$

CLEAR $*$'S USING COLUMN OPS. $\square$

FINAL THING: LET $L^{<0}GL_n$ DENOTE THE KERNEL OF

$$\begin{aligned} L^{-}GL_n &\longrightarrow GL_n \\ g &\longmapsto g \pmod{t^{-1}} \end{aligned}$$

<u>FACT</u> THE MULTIPLICATION MAP

$$L^{<0}GL_n \times L^{+}GL_n \longrightarrow LGL_n$$

IS AN OPEN IMMERSION.

"BIG CELL LEMMA"

(COMPARE: $U^{-} \times B \longrightarrow GL_n$ IS AN OPEN IMMERSION)

<u>IDEA</u> LOOK AT LIE ALGS (AND USE

$$L^{<0}GL_n \cap L^{+}GL_n = \{1\} \ )$$

$\square$

WE NOW DEFINE OPPOSITE SCHUBERT VARS

DEF LET $\mu \in \mathbb{Z}_+^n$. DEFINE

$$Gr^{\mu} = L^{-}Gl_n \cdot t^{\mu} \subset Gr$$ "OPPOSITE SCHUBERT CELL"

$$Gr^{\geq \mu} = \bigsqcup_{\lambda \geq \mu} Gr^{\lambda}$$ "OPP. SCHUBERT VARIETY"

THESE BEHAVE IN A MANNER "DUAL" TO Gr_{μ} (INFINITE DIM FINITE CODIM)

PROP 1) Gr^{μ} IS A LOCALLY CLOSED SUB-IND-SCHEME OF Gr

2) $Gr_{\mu} \cap Gr^{\lambda} \neq \emptyset \iff \mu \geq \lambda$

3) $Gr_{\mu} \cap Gr^{\mu} = G \cdot t^{\mu} \cong G/P_{\mu}$

"CONSTANT LOOPS" (NO t OR t^{-1})

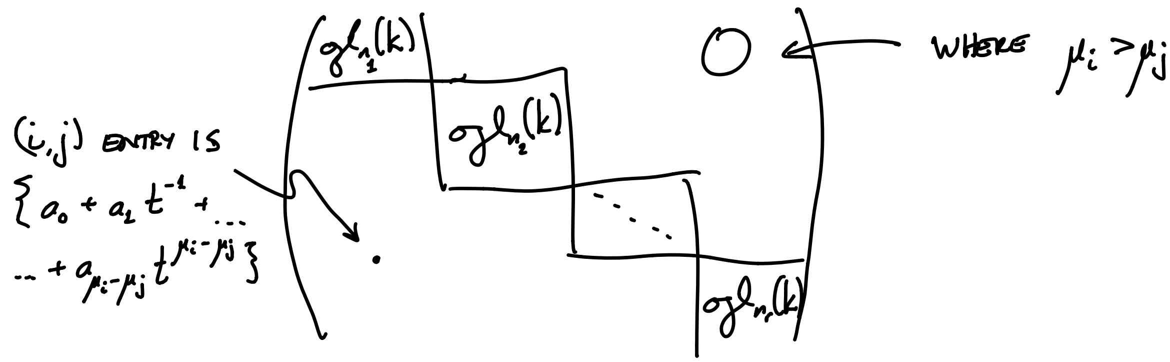
4) $Gr^{\geq \mu}$ IS ZARISKI CLOSED, AND $Gr^{\mu} \subset Gr^{\geq \mu}$ IS OPEN + DENSE

5) LET $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}_+^n$, THEN

$$\mathrm{CODIM}(Gr^{\geq \mu}) = (n-1)\mu_1 + (n-3)\mu_2 + \ldots + (1-n)\mu_n - \dim(G/P_{\mu})$$

PF 1) SIMILAR TO GR_μ: STAB IN L^-GL_n OF $t^\mu \in \mathrm{GR}$ IS $L^-GL_n \cap t^\mu L^+GL_n t^{-\mu}$.

TGT SPACE LOOKS LIKE



$\Rightarrow L^-GL_n \cap t^\mu L^+GL_n t^{-\mu}$ HAS DIM $(n-1)\mu_1 + (n-3)\mu_2 + \dots + \mathrm{DIM}(P_\mu)$

WRITE $L^-GL_n = \operatorname{colim}_i K_i$, WHERE K_i IS FINITE TYPE AND SATISFIES $K_i \supset L^-GL_n \cap t^\mu L^+GL_n t^{-\mu}$. THEN $K_i \cdot t^\mu \subset \mathrm{GR}$ IS LOC. CLOSED AND

$$\mathrm{GR}^\mu = \operatorname{colim}_i K_i \cdot t^\mu \text{ IS AN IND-SCHEME.}$$

4) LOOK AT GR^0. BY 2), $\mathrm{GR}^0 \subset$ CONN. COMP. OF $e = t^0$. FURTHER, WE HAVE $\mathrm{GR}^0 = L^-GL_n \cdot e$
$= L^{<0}GL_n \cdot e$

BY BIG CELL LEMMA, GR^0 IS OPEN IN GR

FIX $\nu \in \mathbb{Z}^n_+$ AND CONSIDER $A_\nu := L^-GL_n \cdot t^\nu \mathrm{GR}^0 \subset \mathrm{GR}$.

A_ν IS OPEN + L^-GL_n - STABLE

$\Rightarrow A_\nu = \bigsqcup_\lambda \mathrm{GR}^\lambda$ FOR SOME λs

"EASY TO SEE" THAT

$$\mathrm{Gr}^\lambda \subset A_\nu \iff \mathrm{Gr}^\lambda \cap \mathrm{Gr}_\nu \neq \emptyset$$
$$\overset{2)}{\iff} \lambda \leq \nu$$

$$A_\nu = \coprod_{\lambda \leq \nu} \mathrm{Gr}^\lambda \quad \text{OPEN}$$

TAKING UNION OVER $\nu \ngeq \mu$ + TAKING COMPLEMENT SHOWS $\coprod_{\lambda \geq \mu} \mathrm{Gr}^\lambda = \mathrm{Gr}^{\geq \mu}$ IS CLOSED AND

$\mathrm{Gr}^\mu \subset \mathrm{Gr}^{\geq \mu}$ IS OPEN

TO SHOW Gr^λ IS IN THE CLOSURE OF Gr^μ, CAN USE A SIMILAR ARGUMENT AS FOR Gr_μ (USING $\mathbb{P}^1$'s, ETC) $\square$

<u>EX</u> (LATTICE DESCRIPTION OF Gr^0)

GIVEN R, DEFINE $\Lambda_0^- := (t^{-1}R[t^{-1}])^n \subset R((t))^n$

THEN Gr^0 REPS THE FUNCTOR

$$R \longmapsto \{ \Lambda \in \mathrm{Gr}(R) : \Lambda_0^- \oplus \Lambda \cong R((t))^n \text{ AS } R\text{-MODS} \}$$

VECTOR BUNDLES

BIRKHOFF DECOMP / OPP. SCHUBERT VARS ARE BETTER UNDERSTOOD USING THE FOLLOWING P.O.V.

LET $\mathrm{Bun}_n(\mathbb{P}^1)$ = MODULI STACK OF RK n V.B's ON $\mathbb{P}^1$ (ALG STACK LOCALLY OF FINITE PRES $/k$)

ALREADY SAW BL MODULI INTERPRETATION:

GR REPS THE FUNCTOR

$$R \longmapsto \left\{ (\mathcal{E}, \beta) : \begin{array}{l} \mathcal{E} \text{ IS A RK } n \text{ V.B ON } \mathbb{P}^1_R \\ \beta : \mathcal{E}|_{\mathbb{P}^1_R \setminus \{0\}} \xrightarrow{\sim} \mathcal{E}^0|_{\mathbb{P}^1_R \setminus \{0\}} \end{array} \right\}$$

GET A MAP TO $\mathrm{BUN}_n(\mathbb{P}^1)$ BY FORGETTING β

$\Longleftrightarrow$ MODDING OUT BY $\mathrm{AUT}(\mathcal{E}^0|_{\mathbb{P}^1 \setminus \{0\}})$

$\cong GL_n(R[t^{-1}])$

<u>THM</u> $\exists$ A CANONICAL ISOM

$$[L^- GL_n \backslash GR] \cong \mathrm{BUN}_n(\mathbb{P}^1)$$

THE MAP $GR \longrightarrow \mathrm{BUN}_n(\mathbb{P}^1)$ EXHIBITS GR AS A $L^- GL_n$ TORSOR IN THE ÉTALE TOP.

<u>IDEA OF PF</u> NEED TO SHOW THAT GIVEN ANY RK n VB $\mathcal{E}$ ON $\mathbb{P}^1_R$, $\exists$ AN ÉTALE EXT $R \longrightarrow R'$ ST $\mathcal{E}_{R'}$ IS TRIVIAL ON $\mathbb{P}^1_{R'} \setminus \{0\}$

$\mathcal{E}|_{\mathbb{P}^1_R \setminus \{0\}} \Longleftrightarrow$ F.G. PROJ $R[t^{-1}]$-MOD M

M/t^{-1} IS F.G. PROJ $/R$ $\Rightarrow$ $\exists$ ETALE EXT $R \to R'$ S.T.

$$M/t^{-1} \otimes_R R' \cong R'^n$$

CAN LIFT GENS OF $M/t^{-1} \otimes_R R'$ TO GET

$$M \otimes_{R[t^{-1}]} R'[t^{-1}] \cong R'[t^{-1}]^n \qquad \square$$

<u>RMK</u> ON k PTS,

$$\mathrm{Bun}_n(\mathbb{P}^1)(k) = L^- GL_n(k) \backslash LGL_n(k) / L^+ GL_n(k)$$

$$\longleftrightarrow \mathbb{Z}^n_+$$

$$\mathcal{O}(\mu) = \bigoplus_{i=1}^n \mathcal{O}(\mu_i) \longleftrightarrow \mu = (\mu_1, \mu_2, \dots, \mu_n)$$

SO, BIRKHOFF DECOMP $\longleftrightarrow$ GROTHENDIECK'S THM ON VB'S ON $\mathbb{P}^1$

GEOMETRY OF THE $\mathrm{Gr}^{\geq \mu}$ TELLS US ABOUT GEOMETRY OF $\mathrm{Bun}_n(\mathbb{P}^1)$

<u>EX</u> $\mathrm{Gr}^{(1,-1)} \subset \mathrm{Gr}^{\geq 0}$ IMPLIES $\exists$ DEGENERATION OF RK 2 VBS $\mathcal{O} \oplus \mathcal{O} \rightsquigarrow \mathcal{O}(1) \oplus \mathcal{O}(-1)$

OTHER STATEMENTS OF THIS TYPE:

- $[\mathcal{O}(\lambda)]$ IS IN THE CLOSURE OF $[\mathcal{O}(\mu)]$ $\Longleftrightarrow$ $\lambda \geq \mu$
- CLOSURE OF $[\mathcal{O}(\mu)]$ IS A CLOSED SUBSTACK OF DIM $-\langle 2\rho, \mu \rangle - \dim(P_\mu)$

PICARD GROUPS

TODAY: $G = SL_n$

$Gr' = Gr_{SL_n}$:

$$R \longmapsto \left\{ \Lambda \subset R((t))^n \text{ F.G. PROJ } R[[t]]\text{-MODS S.T. } \begin{array}{l} \cdot\ \Lambda[1/t] = R((t))^n \\ \cdot\ \det_{R[[t]]}(\Lambda) = R[[t]] \end{array} \right\}$$

FACTS

- $Gr' \cong [LSL_n / L^+SL_n]$
- Gr' IS REDUCED, AND $Gr' \cong$ CONN. COMP. OF id IN $(Gr)_{RED}$
- GEOMETRY OF (OPP) SCHUBERT VARS BEHAVES AS FOR GL_n, w/ $\mathbb{Z}^n_+$ REPLACED BY $\mathbb{Z}^n_{+,0} := \{ \mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n_+ : \sum_i \mu_i = 0 \}$

RECALL: Gr' HAS A VERY AMPLE DETERMINANT LINE BUNDLE $\mathcal{L}$:

GIVEN $\Lambda \in Gr'(R)$, SET

$$\det_R(\Lambda)^{-1} := \det_R(\Lambda_0/t^N\Lambda_0) \otimes \det_R(\Lambda/t^N\Lambda_0)^{-1}$$
$$\in \mathrm{Pic}(R) \qquad (\text{FOR } N \gg 0)$$

(HERE: $\Lambda_0 = R[\![t]\!]^n$)

DEFINE $\Lambda_0^- := (t^{-1}R[t^{-1}])^n \subset R(\!(t)\!)^n$

NOTE: $R(\!(t)\!)^n = \Lambda_0 \oplus \Lambda_0^-$ AND $\mathrm{STAB}_{LSL_n}(\Lambda_0^-) = L^-SL_n$

CONSIDER THE FOLLOWING COMPLEX (FOR A FIXED $\Lambda \in Gr'(R)$):

DEGREE: $\quad -1 \qquad\qquad 0$

$$0 \longrightarrow \underbrace{\Lambda_0^- \oplus \Lambda}_{\text{ABSTRACT } \oplus} \xrightarrow{\gamma_\Lambda} \underbrace{R(\!(t)\!)^n = \Lambda_0^- \oplus \Lambda_0}_{\text{INTERNAL } \oplus} \longrightarrow 0$$

WHERE $\gamma_\Lambda(v,w) = v+w$.

THE DETERMINANT OF THIS COMPLEX (I.E. $\bigotimes_i \det_R(W_i)^{\otimes(-1)^i}$) IS EXACTLY $\mathcal{L} = \det_R(\Lambda)^{-1}$,

WHERE WE MOD OUT BY $\Lambda_0^- \oplus t^N\Lambda_0$ FOR $N \gg 0$

DEFINE σ BY THE FOLLOWING RECIPE :

TAKE $N \gg 0$, MOD OUT $\Lambda_0^- \otimes t^N \Lambda_0$ TO GET

$$0 \longrightarrow \Lambda / t^N \Lambda_0 \xrightarrow{\overline{\gamma}_\Lambda} \Lambda_0 / t^N \Lambda_0 \longrightarrow 0$$

SET $\boxed{\sigma(\Lambda) := \det_R(\overline{\gamma}_\Lambda)} \in \det_R(\Lambda)^{-1}$

THE MAP σ DEFINES A GLOBAL SECTION OF $\mathcal{L}$

ROUGH IDEA: $\sigma(\Lambda) \neq 0$ IFF Λ AND Λ_0^- DON'T INTERSECT

EX $R = k$ $\quad \sigma(\Lambda_0) = 1$

$$\sigma\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} . \Lambda_0\right) = 0$$

IN GENERAL : LET $g \in LSL_n(R)$. THEN

$$\sigma(g\Lambda_0) = 0 \iff g\Lambda_0 \cap \Lambda_0^- \neq 0$$

$$\iff g \notin L^- SL_n(R) \cdot L^+ SL_n(R)$$

$$\iff g\Lambda_0 \notin Gr'^{0} = L^- SL_n \cdot R$$

RECALL THE COROOT $\theta = (1, 0, 0, \dots, 0, -1) \in \mathbb{Z}^n_{+,0}$.

THE UNIQUE ELT OF $\mathbb{Z}^n_{+,0}$ WHICH IS $< \theta$ IS 0

$\Longrightarrow$ WE HAVE

$$Gr' = Gr'^{\geq 0} = Gr'^{\geq \theta} \sqcup Gr'^{0}$$

DEFINE $\boxed{\Theta := Gr^{\geq \theta}}$

KNOW Θ IS EXACTLY THE VANISHING SET OF σ, AND $\mathrm{codim}(\Theta, Gr') = 1$ (FROM LAST TIME)

GIVE Θ REDUCED CLOSED SUB-IND-SCHEME STR. THEN

THM Θ IS AN EFFECTIVE CARTIER DIVISOR, AND $\mathcal{O}(1) := \mathcal{O}(\Theta) = \mathcal{L}$ IS (VERY) AMPLE.

FURTHER, WE HAVE

$$\mathrm{Pic}^e(Gr') = \mathbb{Z}\,\mathcal{O}(1)$$

L.B.'S W/ RIGID'N AT $e = t^0$ (ISOM $\varepsilon : e^*\mathcal{L}' \to k$)

(Q: RIGID'N NECESSARY?)

IDEA PULL BACK FROM $Bun_n'(\mathbb{P}^1)$

DEFINE $Bun_n'(\mathbb{P}^1)$ = MODULI STACK OF RK n V.B'S W/ TRIVIALIZED $\det$

AS BEFORE $[L^-SL_n \backslash Gr'] = Bun_n'(\mathbb{P}^1)$

LET $\mathring{Bun}_n'(\mathbb{P}^1)$ BE THE OPEN SUBSTACK CLASSIFYING V.B'S ON $\mathbb{P}^1$ WHICH ARE TRIVIALIZABLE

THEN

$$[L^-SL_n \backslash Gr'^{\circ}] = \mathring{Bun}_n'(\mathbb{P}^1)$$

$$[L^-SL_n \backslash \Theta] = Bun_n'(\mathbb{P}^1) \setminus \mathring{Bun}_n'(\mathbb{P}^1)$$

AND $[L^{-}SL_n \setminus \Theta]$ IS PURE OF CODIM 1

$\implies$ THIS DEFINES AN EFFECTIVE CARTIER DIV AND A LINE BUNDLE $\mathcal{O}(1)$ ON $\mathrm{BUN}_n'(\mathbb{P}^1)$

PULLING BACK THIS $\mathcal{O}(1)$ VIA $GR' \longrightarrow \mathrm{BUN}_n'(\mathbb{P}^1)$ GIVES AN AMPLE LINE BUNDLE $\mathcal{L}(\Theta)$ $(\cong \mathcal{L})$

RECALL : WHEN COMPUTING CLOSURE RELS OF (OPP) SCHUBERT VARS, WE HAD A MAP

$$\varphi : \mathbb{P}^1 \longrightarrow Gr'$$

ST

$$\varphi(\mathbb{A}^1) \subset Gr'^{\,0}$$
$$\varphi(\infty) = t^{\theta} \in \Theta = Gr'^{\,\geq \theta}$$

AND MOREOVER THIS $\mathbb{P}^1$ INTERSECTS Θ TRANSVERSALLY.

PULLING $\mathcal{L}(\Theta)$ BACK TO $\mathbb{P}^1$ GIVES INJECTIVE MAPS

$$\begin{array}{ccccc} & & ? \rightsquigarrow e & & \\ \mathbb{Z}\mathcal{L}(\Theta) & \hookrightarrow & \mathrm{Pic}(Gr') & \xhookrightarrow{\varphi^*} & \mathrm{Pic}(\mathbb{P}^1) \\ \mathcal{L}(\Theta) & & \longmapsto & & \mathcal{O}_{\mathbb{P}^1}(1) \end{array}$$

$? \rightsquigarrow e$

$\implies \varphi^*$ IS AN ISOM AND $\mathrm{Pic}(Gr') = \mathbb{Z}\mathcal{L}(\Theta)$ $\blacksquare$